AN ECONOMIC ANALYSIS OF ONLINE ADVERTISING USING BEHAVIORAL TARGETING

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A.2 Proof of Lemma 1

Proof. We consider any advertiser $i$’s bidding strategy $b_i$. We suppose that among the rest of advertisers, advertiser $l$ has the highest expected payment $b_i \cdot p_i E[q]$. If $b_i \cdot p_i E[q] > b_l \cdot p_l E[q]$, advertiser $i$ wins the auction with total expected payment $b_i p_i E[q]$, and his payoff is

$$v_i p_i E[q] - b_l p_l E[q]$$

(16)

Otherwise, the advertiser loses the auction and his payoff is zero.

We first consider bidding $b_i < v_i$. If $b_l \cdot p_l E[q] > b_i \cdot p_i E[q]$, the advertiser wins the auction and derives the payoff in Equation (16), just as if bidding $v_i$. If $b_i \cdot p_i E[q] > v_i \cdot p_i E[q]$, the advertiser loses the auction and derives zero payoff, just as if bidding $v_i$. If $b_i \cdot p_i E[q] < b_l \cdot p_l E[q] < v_i \cdot p_i E[q]$, bidding $b_i$ causes the advertiser to lose the auction and receive zero payoff, whereas if he had bid $v_i$, the advertiser would have won the auction and derived positive payoff as in Equation (16). Therefore, bidding $b_i < v_i$ is (weakly) dominated by bidding $v_i$.

We then consider bidding $b_i > v_i$. If $v_i \cdot p_i E[q] > b_i \cdot p_i E[q]$, the advertiser wins the auction and derives the payoff in Equation (16), just as if bidding $v_i$. If $b_i \cdot p_i E[q] > b_l \cdot p_l E[q]$, the advertiser loses the auction and derives zero payoff, just as if bidding $v_i$. If $b_i \cdot p_i E[q] < b_l \cdot p_l E[q] < v_i \cdot p_i E[q]$, bidding $b_i$ causes the advertiser to win the auction with negative payoff, whereas bidding $v_i$ leads to zero payoff. Therefore, bidding $b_i > v_i$ is (weakly) dominated by bidding $v_i$. All together, bidding true value $v_i$ is advertiser $i$’s (weakly) dominant strategy.

Under behavioral targeting, the same argument applies, simply replacing $p_i E[q]$ with $p_i q_j$ for user $j$.

A.3 Proof of Lemma 2

Proof. For any advertiser $k$ different from advertisers $i$ and $i + 1$, we first consider the case where the shortest path from user $j$ (located between advertisers $i$ and $i + 1$) to advertiser $k$ passes advertiser $i$. If $z_i \geq z_j$, then $z_i (1 - \gamma_{ij}) < z_i (1 - \gamma_{ij})$ because $x_{ij} < x_{ij}$. If $z_i < z_i$, we have (assuming $k < i$)

$$z_k (1 - \gamma_{ij}) = z_i \left(1 - \frac{i-k}{n} + x_{ij}\right) < z_k \gamma_{ij} < z_i (1 - \gamma_{ij})$$

where the first inequality follows from the comparable value assumption and the second inequality is because $z_i < z_i$. Therefore, in this case, advertiser $i$ derives higher value than advertiser $k$ from user $j$. Similarly, when the shortest path from user $j$ to advertiser $k$ passes advertiser $i + 1$, advertiser $i + 1$ derives higher value than advertiser $k$ from user $j$.

All together, either advertiser $i$ or advertiser $i + 1$ derives higher value than any other advertiser $k$ from user $j$, and thus user $j$ must be assigned to either advertiser $i$ or advertiser $i + 1$ in equilibrium.

A.4 Proof of Lemma 3

Proof. We denote $V_y = z_j q_j$, the value that the advertiser $i$ derives from user $j$, and denote $V_{(1)}$ and $V_{(2)}$ as the highest and second highest values among all $i$’s. When $z_i$ is increased to $z_i$ (i.e., $z_i > z_i$), $V_{(1)} > V_y$ and thus $V_{(2)} \geq V_{(1)}$. In addition, for the additional users that advertiser $i$ wins under $z_i$ the highest values under $z_i$ become the second highest ones under $z_i$, and thus $V_{(2)} = V_{(1)} > V_{(2)}$. Because the publisher’s revenue is the sum of the second highest value from each user, $\pi_y > \pi_x$.

A.5 Proof of Proposition 1

Proof. (a) By Lemma 3, $\partial \pi_y / \partial z_i > 0$. Given $z_{i0}$ and the order constraint $z_{i0} \geq z_{(i+1)}$, to maximize $\pi_y$, $z_{i0} = z_{i1} = \ldots = z_{i0} = z_{i1}$. For $z_{(1)}$, we have $\partial \pi_y / \partial z_{(1)} > 0$ subject to $z_{(1)} \geq z_{(1)} \left(1 - \frac{2}{n}\right)$ (i.e., the comparable value assumption). Noticing $z_{i0} = z_{i1}$, we conclude the $z_{i0}$ that maximizes $\pi_y$ is $z_{i0} = z_{i1} \left(1 - \frac{2}{n}\right)$ (when the condition binds).
(b) Similarly, because \( \partial \pi_\theta / \partial z_i > 0 \), to minimize \( \pi_\theta \) subject to \( z_1 \) and \( z_2 \geq z_\theta \) we have \( z_1 = z_2 \) and \( z_3 = z_4 = \cdots = z_{\theta-1} \). Because of the comparable value assumption \( z_\theta \geq z_1 \left( 1 - \frac{Y}{n} \right) \), the \( z_\theta \) that minimizes \( \pi_\theta \) is \( z_\theta = \left( 1 - \frac{Y}{n} \right) \).

We next determine how the relative location of the two advertisers with \( z_2 \) affects the revenue. When \( n = 2 \) or 3, there is a unique relative distribution of these advertisers’ values. We next focus on the case with \( n \geq 4 \).

We first examine the possible revenue for a user segment \( i \) \( \mid (i + 1) \). For a user segment with \( z_i = z_2 \) and \( z_{i+1} = z_2 \), the revenue is from a base payment (in Table A1) by Equation (7) with \( x_i = 1/2n \). For user segment \( z_2 \) \( \mid z_{\theta} \), if \( z_2 \) ’s other neighbor is of \( z_{\theta} \) the revenue is from a base payment (in Table A1) by Equation (7) with \( x_i = 1/n \); if \( z_2 \) ’s other neighbor is of \( z_2 \), the revenue is the base payment plus the cross-border effect.

By Equation (10), the cross-border effect is

\[
\frac{\left(1 - \frac{Y}{n}\right)^2 \left(z_2 - z_\theta\right)^2}{2}\left(1 - \frac{Y}{n}\right)(1 - \frac{Y}{2n}) = \frac{Y(n - Y)^2}{2n^2(2n - Y)} z_2
\]

(17)

For user segment \( z_\theta \) \( \mid z_{\theta} \), if the neighbors are of \( z_{\theta} \), the revenue is from a base payment (in Table A1) by Equation (7) with \( x_i = 1/(2n) \); if one neighbor is of \( z_2 \) and the other is of \( z_{\theta} \), the revenue is the base payment plus one piece of the cross-border effect defined in (17). If both neighbors are of \( z_2 \), the revenue is the base payment plus two pieces of the cross-border effect (with one piece at each border).

| Table A1. The Base Payments for Possible User Segments |
|-----------------------------|-----------------------------|-----------------------------|
| User Segment | \( z_2 \) | \( z_2 \) | \( z_2 \) |
| Base Payment | \( \frac{1}{n} \left( 1 - \frac{3Y}{4n} \right) z_2 \) | \( \frac{1}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{Y}{2n}) z_2 \) | \( \frac{1}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{3Y}{4n}) z_2 \) |

Notice the base payment from each user segment \( i \) \( \mid (i + 1) \) is solely determined by the values \( z_i \) and \( z_{i+1} \) (while the cross-border effect depends on the neighbors’ values). Therefore, the revenue from base payments (excluding the cross-border effect) can differ only when the two highest-value advertisers are adjacent and when they are not. In the former, the revenue consists revenue from one \( z_2 \) \( \mid z_2 \) segment, from two \( z_2 \) \( \mid z_\theta \) segments, and from \( n - 3 \) \( z_\theta \) \( \mid z_\theta \) segments; in the latter (the non-adjacent case), the revenue consists of revenue from four \( z_2 \) \( \mid z_\theta \) segments and from \( n - 4 \) \( z_\theta \) \( \mid z_\theta \) segments. The difference in these revenues is the revenue from one \( z_2 \) \( \mid z_2 \) and from one \( z_\theta \) \( \mid z_\theta \) minus the revenue from two \( z_2 \) \( \mid z_\theta \) that is,

\[
\frac{1}{n} \left( 1 - \frac{3Y}{4n} \right) z_2 + \frac{1}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{3Y}{4n}) z_2 - \frac{2}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{Y}{2n}) z_2 = \frac{1}{n} \left( 1 - \frac{3Y}{4n} \right) z_2 = \frac{1}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{3Y}{4n}) z_2 = \frac{1}{n} \left( 1 - \frac{Y}{n} \right)(1 - \frac{Y}{2n}) z_2 > 0
\]

which indicates that the revenue from base payments is greater when the highest-value advertisers are adjacent.

If the two highest-value advertisers are adjacent, the total revenue includes four pieces of cross-border effects, in addition to the revenue from the base payment. When \( n = 4 \), the four pieces consist of two from the two \( z_2 \) \( \mid z_2 \) segments (because in each segment \( z_2 \) neighbors the other \( z_2 \) and two from the \( z_\theta \) \( \mid z_\theta \) segment (because both \( z_\theta \)’s neighbor \( z_2 \)). When \( n \geq 5 \), the four pieces consist of two from the two \( z_2 \) \( \mid z_2 \) segments and two from the two \( z_\theta \) \( \mid z_\theta \) segments with one \( z_\theta \) in each segment neighbored by \( z_2 \).

If the two highest-value advertisers are not adjacent, the total revenue contains no cross-border effect when \( n = 4 \). Therefore, the total revenue (including the revenue from the base payment and the cross-border effect) is lower when the two highest-value advertisers are not adjacent than when they are. When \( n \geq 5 \), if the two highest-value advertisers are \( 2/n \) distant to each other (i.e., there is one lowest-value advertiser in between), the total revenue contains two pieces of cross-border effect: either from the two \( z_\theta \) \( \mid z_\theta \) segments with one \( z_\theta \) neighbored by \( z_2 \) (if \( n > 5 \)) or from the \( z_\theta \) \( \mid z_\theta \) segment with both \( z_\theta \)’s neighbored by \( z_2 \) (if \( n = 5 \)). If at least two lowest-value advertisers are located between the two highest-value advertisers, the total revenue contains four pieces of cross-border effect. The reason is that, around each arc connecting the two \( z_2 \)’s, we should have either two \( z_\theta \) \( \mid z_\theta \) segments with one \( z_\theta \) neighbored by \( z_2 \) in each segment or one \( z_\theta \) \( \mid z_\theta \) segment with both \( z_\theta \)’s neighbored by \( z_2 \). Either case leads to two pieces of cross-border effect with one arc, and we have two different arcs. Therefore, when \( n \geq 5 \), the structure with the two highest-value advertisers being \( 2/n \) distant from each other generates the least total revenue (with the lowest revenue from the base payment and the least number of cross-border effects).
A.6 Proof of Proposition 2

**Proof.**
(a) When \( n = 2 \), by Proposition 1, given \( z_{(2)} \), the structure with \( z_{(i)} = z_{(2)}/(1 - \gamma/2) \) generates the highest \( \pi_b \), in which the dominant advertiser wins all users (except the other advertiser’s most targeted user, from which both advertisers derive the same value). In this case, by Equation (7) with \( x_i = 1/2 \), \( \pi_b = \left(1 - \frac{\gamma}{4}\right)z_{(2)} \), which is the same as \( \pi_T \) by Equation (3). Therefore, \( \pi_b \leq \pi_T \) and the equality occurs only if \( z_{(i)} = z_{(2)}/(1 - \gamma/2) \).

When \( n \geq 3 \), \( \pi_b > \pi_T \) is always possible. For example, when \( z_{(i)} = z_{(2)} = \ldots = z_{(n)} \), \( \pi_b \) consists of the base payments from the \( n \) \( z_{(2)} \) | \( z_{(n)} \) segments. According to Table A1, \( \pi_b = \left(1 - \frac{3\gamma}{4n}\right)z_{(2)} \). Noticing \( \pi_T = \left(1 - \frac{\gamma}{4}\right)z_{(2)} \), we can conclude \( \pi_b \geq \pi_T \) when \( n \geq 3 \). Furthermore, a value structure with \( z_{(n)} > z_{(2)} \) results in a higher revenue under behavioral targeting (by Lemma 3) and in the same revenue under traditional advertising, which leads to \( \pi_b \geq \pi_T \).

(b) For \( 2 < n < 6 \), traditional advertising might generate higher revenue than behavioral targeting as well. For \( n = 5 \), see the proof in part (c). When \( n = 4 \), the least revenue under behavioral advertising is \( \left(1 - \frac{2\gamma}{3}\right)\left(1 - \frac{\gamma}{8}\right)z_{(2)} \) (from four \( z_{(2)} \mid z_{(n)} \) segments), which is less than \( \pi_T = \left(1 - \frac{\gamma}{4}\right)z_{(2)} \), regardless of \( \gamma \). When \( n = 3 \), the least revenue consists of the base payment (from two \( z_{(2)} \mid z_{(n)} \) segments and one \( z_{(2)} \mid z_{(2)} \) segment) and two pieces of cross-border effect:

\[
\frac{1}{3}\left(1 - \frac{\gamma}{4}\right)z_{(2)} + \frac{2}{3}\left(1 - \frac{\gamma}{3}\right)\left(1 - \frac{\gamma}{6}\right)z_{(2)} + \frac{\gamma(3 - \gamma)^2}{27(6 - \gamma)}z_{(2)} = \left(1 - \frac{\gamma}{4}\right)z_{(2)} + \left[\frac{1}{6} + \frac{\gamma}{27} + \frac{(3 - \gamma)^2}{3(6 - \gamma)}\right]z_{(2)}
\]

which is less than \( \pi_T \) regardless of \( \gamma \) because the term in the square bracket is negative.

(c) When \( n \geq 5 \), according to the proof of Proposition 1, the least revenue consists of the base payment (from four \( z_{(2)} \mid z_{(n)} \) segments and \( n - 4 \) \( z_{(n)} \mid z_{(n)} \) segments) and two pieces of cross-border effect:

\[
\pi_b = \frac{4}{n}\left(1 - \frac{\gamma}{n}\right)\left(1 - \frac{\gamma}{2n}\right)z_{(2)} + \frac{n - 4}{n}\left(1 - \frac{\gamma}{n}\right)\left(1 - \frac{3\gamma}{4n}\right)z_{(2)} + \frac{\gamma(n - \gamma)^2}{n^2(2n - \gamma)}z_{(2)}
\]

The difference between this \( \pi_b \) and \( \pi_T \) in Equation (3) is

\[
\left[\left(1 - \frac{\gamma}{n}\right)\left(1 - \frac{3n - 4}{4n^2}\right) + \frac{\gamma(n - \gamma)^2}{n^2(2n - \gamma)}\right]z_{(2)} = \left[n^2(n - 4) - (3n - 4)(n - \gamma) + \frac{4(n - \gamma)^2}{2n - \gamma}\right]z_{(2)}
\]

The term in the second square bracket is increasing in \( \gamma \) by noting the first-order derivative

\[
3n - 4 - \frac{8(n - \gamma)}{2n - \gamma} + \frac{4(n - \gamma)^2}{(2n - \gamma)^2} > 3n - 4 - \frac{8(n - \gamma)}{2n - \gamma} = 3n - 8 + \frac{4\gamma}{2n - \gamma} > 0
\]

and thus is greater than its value at zero \( n^2(n - 4) - (3n - 4)n + 2n = n(n - 6)(n - 1) \) (which is nonnegative if \( n \geq 6 \)). Therefore, the difference is positive, and behavioral advertising generates higher revenue if \( n \geq 6 \). If \( n = 5 \), the difference is negative for any \( \gamma \) by noting that the value of the term in the square bracket at \( \gamma = 2 \) is \(-3.5\). All together, we can conclude that if \( n \leq 5 \), the revenue under behavioral targeting may be less than the revenue under traditional advertising. If \( n \geq 6 \), the least amount of revenue under behavioral targeting is still greater than the revenue under traditional advertising. Therefore, if and only if \( n \geq 6 \), the publisher is better off using behavioral targeting.

A.7 Proof of Corollary 1

**Proof.** Under behavioral targeting, \( x_i = \frac{1}{2n} \) by Equation (6). According to Equation (13), \( \pi_b = \left(1 - \frac{3\gamma}{4n}\right) \) by substituting in \( x_i \) and noting \( \Delta_i = 0 \).

According to Equation (3), \( \pi_T = \left(1 - \frac{\gamma}{4}\right) \). Therefore, when \( n = 3 \), \( \pi_b = \pi_T \), and when \( n > 3 \), \( \pi_b > \pi_T \).
A.8 Proof of Proposition 3

Proof. (a) When \( n = 2 \), according to Proposition 2, the publisher is (weakly) better off under traditional advertising. Meanwhile, when the lower-value advertiser gets no market share under behavioral targeting, two advertising strategies could lead to the same revenue for the publisher. Therefore, if \( n = 2 \), the maximum gain is zero.

When \( n > 2 \), without loss of generality, we let \( z_1 = z_{(1)} \), and normalize \( z_{(2)} = 1 \). \( \pi_B \) is maximized when \( z_{(2)} = z_{(3)} = \ldots = z_{(n)} = 1 \) and \( z_{(i)} = \frac{1}{n} \left( 1 - \frac{\gamma}{n} \right) \)

by Proposition 1. By substituting \( z_{(i)} \) and \( x_i = 1/n \) into (14), we can obtain the maximum \( \pi_B \) and thus calculate the maximum gain as

\[
\frac{\pi_B}{\pi_T} = 1 - \frac{n - 2}{n} \left( \frac{1}{4n} \right)^2 + \frac{2}{n} \left( \frac{1 - \frac{\gamma}{n}}{2} \right)^2 + \frac{(n - \gamma)(n - 2)}{4n} \gamma \left( \frac{1}{n} \right)^2 - 1 = \frac{1 - \frac{\gamma}{n}}{4} - \frac{1 - \frac{\gamma}{n}}{1 - \frac{\gamma}{n}} \gamma 
\]

(18)

(b) We notice the first-order derivative of the term in the square bracket on the right-hand side of (18) with respect to \( n \),

\[
\frac{3n - 4}{n} - 2 \frac{2(n - \gamma)}{n(2n - \gamma)} - 2 \frac{2(n - \gamma)^2 - 4n^2 \gamma + \gamma^2}{(2n - \gamma)^2 n^2} > \frac{3n - 4}{n^2} - 2 \frac{2n^2 - 4n^2 \gamma + \gamma^2}{(2n - \gamma)^2 n^2} = \frac{(2n - 4)(2n - \gamma)^2 + (4n^2 \gamma - n \gamma^2)}{(2n - \gamma)^2 n^2} > 0
\]

which indicates the maximum gain is increasing in \( n \).

We notice the first-order derivative of the right-hand side of (18) with respect to \( \gamma \),

\[
\frac{4}{(4 - \gamma)^2} \left[ \frac{(n - 2)(n - 1)}{n^2} + \frac{2(n - \gamma)}{n(2n - \gamma)} \right] - \frac{1}{2} \frac{2(n - \gamma)}{(2n - \gamma)^2 n^2} = \frac{8(n - \gamma)}{(4 - \gamma)^2 (2n - \gamma)^2 n^2} + \frac{(n - 2)(n - 1)(2n - \gamma)}{(2n - \gamma) n^2} + 2(n - \gamma)(2n - \gamma) - (4 - \gamma)^2 n \gamma
\]

Note that the term in the above square bracket is greater than \( (2n - \gamma)^2 + 2(n - \gamma)(2n - \gamma) - (4 - \gamma)^2 n \gamma = 8n^2 - 14n \gamma + 3 \gamma^2 + n \gamma^2 \), which is positive. Therefore, the above first-order derivative is positive, and the maximum gain is increasing in \( \gamma \).

A.9 Proof of Proposition 4

Proof. Part (a) can be found to be true from the discussion in the body.

(b) We first derive advertiser \( i \)'s payoff under behavioral targeting:

\[
\left[ z_i (1 - \gamma x) - z_{i+1} \left( 1 - \frac{\gamma}{n} + \gamma x \right) \right] dx + \left[ z_i (1 - \gamma x) - z_{i-1} \left( 1 - \frac{\gamma}{n} + \gamma x \right) \right] dx = \Delta_i
\]

\[
= z_i - z_{i-1} \left( \frac{1 - \gamma}{n} \right) x_i - \frac{\gamma}{2} (z_i + z_{i+1}) x_i^2 + z_i - z_{i-1} \left( 1 - \frac{\gamma}{n} \right) \left( z_i - x_i \right) - \frac{\gamma}{2} (z_i + z_{i+1}) \left( 1 - \frac{x_i}{n} \right)^2 = \Delta_i
\]

(19)

where the first integral is the expected value net the base payment for the users in segment \( i \mid (i + 1) \), the second integral is the expected value net the base payment for the users in segment \( (i - 1) \mid i \), \( \Delta_i \) is the cross-border effect, and the last equality is because
\[\gamma(z_i + z_{i-1})x_i = z_i - z_{i-1}\left(1 - \frac{Y}{n}\right)\]
\[\gamma(z_i + z_{i-1}\left(1 - x_{i-1}\right)) = z_i - z_{i-1}\left(1 - \frac{Y}{n}\right)\]
from Equation (6).

(b.1) When \(n = 2\), the difference in the payoff under behavioral targeting and under traditional advertising is

\[
\frac{\left(z_i - z_j\left(1 - \frac{Y}{n}\right)\right)^2}{\gamma(z_i + z_{j})} - (z_i - z_j)\left(1 - \frac{Y}{4}\right) = \frac{1}{\gamma(z_i + z_{j})}\left[\left(1 - \frac{Y}{2}\right)^2 z_i^2 - 2\left(1 - \frac{Y}{2}\right)z_i z_j + z_j^2\right] = \frac{1}{\gamma(z_i + z_{j})}\left(1 - \frac{Y}{2}\right)z_i - z_j\right]^2 \geq 0
\]
where the first term (the advertiser’s payoff under behavioral targeting) is from Equation (19) (noticing \(\Delta_i = 0\)).

(b.2) Advertiser 1’s payoff under behavioral targeting can be formulated by letting \(i = 1\) in Equation (19), and the payoff under traditional advertising is \((z_{i-1} - z_i)\left(1 - \frac{Y}{4}\right)\) by Equation (4). The condition for advertiser 1 to be better off under behavioral targeting specified in the proposition then follows.

A.10 Proof of Corollary 4

Proof. The difference in the payoff under behavioral targeting and under traditional advertising is

\[
\Delta(z_i) = \frac{\left(z_i - \left(1 - \frac{Y}{n}\right)\right)^2}{\gamma(z_i + 1)} -(z_i - 1)\left(1 - \frac{Y}{4}\right) = \frac{1}{\gamma(z_i + 1)}\left[\left(1 - \frac{Y}{2}\right)^2 z_i^2 - 2\left(1 - \frac{Y}{2}\right)z_i z_j + z_j^2\right] = \frac{1}{\gamma(z_i + z_j)}\left(1 - \frac{Y}{2}\right)z_i - z_j\right]^2 \geq 0
\]
\(\Delta(z_i) = 0\) has two roots:

\[
z_i^* = \frac{\left(1 - \frac{Y}{n}\right) - \sqrt{\left(1 - \frac{Y}{4}\right)\left(\left(1 - \frac{Y}{n}\right)^2 - \left(1 - \frac{Y}{2}\right)^2\right)}}{-\left(1 - \frac{Y}{2}\right)^2} \quad \text{and} \quad z_j^* = \frac{\left(1 - \frac{Y}{n}\right) + \sqrt{\left(1 - \frac{Y}{4}\right)\left(\left(1 - \frac{Y}{n}\right)^2 - \left(1 - \frac{Y}{2}\right)^2\right)}}{-\left(1 - \frac{Y}{2}\right)^2}
\]
with \(z_i < z_j^*\). Notice that \(\Delta(1) > 0\) because when \(z_i = 1\) the advertiser’s payoff under behavioral targeting is positive and his payoff under traditional advertising is zero. Furthermore, because \(z_i + z_j^* = 2\left(1 - \frac{Y}{n}\right)/\left(1 - \frac{Y}{2}\right)^2 > 2\), we have \(z_i > 1\). Under the comparable value assumption, \(z_i < 1/\left(1 - \frac{Y}{n}\right)\). Because \(1/\left(1 - \frac{Y}{2}\right)^2 < 1/\left(1 - \frac{Y}{n}\right)^2\) \(z_i < z_j^*\), it follows that \(z_i < z_j^*\). Therefore, if \(z_i < z_j^*\), \(\Delta(z_i) > 0\) and thus the advertiser is better off under behavioral targeting; otherwise, it is worse off. Notice that \(z_i \in \left[1, 1/\left(1 - \frac{Y}{n}\right)\right]\). \(z_i < z_i^*\) can occur because \(z_i > 1\). \(z_i > z_j^*\) can occur because

\[
\frac{1}{1 - \frac{Y}{n}} - z_i^* = \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right] \quad \text{and} \quad \frac{1}{1 - \frac{Y}{n}} - z_j^* = \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right]
\]

\[
= \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right] \quad \text{and} \quad \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right]
\]

\[
= \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right] \quad \text{and} \quad \frac{1}{\left(1 - \frac{Y}{n}\right)^2}\left[\left(-\frac{Y}{2}\right)^2 - \left(1 - \frac{Y}{n}\right)^2\right]
\]

\[
> 0
\]
A.11  Proof of Proposition 5

Proof. We assume that advertiser 1 is the dominant advertiser. The joint payoff of the publisher and the advertisers under traditional advertising can be formulated as $2z_1 \int_0^1 (1 - \gamma x) \, dx$. The joint payoff under behavioral targeting is $\sum_{i=1}^n \int_0^{x_i} z_i (1 - \gamma x) \, dx + \int_{x_i}^{x_{i+1}} z_i (1 - \gamma x) \, dx$.

For any user $j$ in segment $i \mid (i + 1)$, if $x_{ij} \in [0, x_i]$, by Lemma 2 and the definition of $x_i$, advertiser $i$ wins the user and derives the highest value among all advertisers, which implies

$$z_i (1 - \gamma x_{ij}) > z_i (1 - \gamma x_{ij})$$

The left-hand side of the inequality is also user $j$’s contribution to the joint payoff under behavioral targeting, and the right-hand side is her contribution to the joint payoff under traditional advertising. Therefore, the joint payoff derived from user $j$ is higher under behavioral targeting than under traditional advertising. The same argument applies if $x_{ij} \in [x_i, 1/n]$ (such that advertiser $i + 1$ wins the user and derives the highest value). Because the joint payoff is the sum of the joint payoff from each user, the joint payoff under behavioral targeting is greater than the joint payoff under traditional advertising.

A.12  Maximum Gain in the General Case

Proof. Without loss of generality, we let $z_1 = z_{(1)}$ and normalize $z_{(2)} = 1$. By Lemma 3, $\pi_\beta$ is maximized when $z_{(2)} = z_{(3)} = \ldots = z_{(n)}$ and when $z_{(1)}$ is large enough to win all of the users. For each of the two segments with one end at $z_{(1)}$ (i.e., segments $n \mid 1$ and $1 \mid 2$), the expected revenue is the lower-value advertiser’s expected value of all the users in the segment, which is $\frac{1}{n} (1 - \frac{\gamma}{2n})$. For each of the other $n - 2$ segments (e.g., $i \mid i + 1$), the expected revenue is advertiser $i$’s expected value of the half of the users who are closer to him than to advertiser $i + 1$ and advertiser $i + 1$’s expected value of the other half, which is $\frac{1}{n} (1 - \frac{\gamma}{4n})$. We can thus obtain the maximum gain and calculate the maximum $\pi_\beta$ gain as

$$\frac{\pi_\beta}{\pi_\gamma} = \frac{n - 2}{n} \left(1 - \frac{\gamma}{4n}\right) + \frac{2}{n} \left(1 - \frac{\gamma}{2n}\right) = \frac{\gamma}{4} - \frac{\gamma (n - 2)}{4n} + 1 = \frac{\gamma (1 - \frac{n + 2}{n^2})}{4 - \gamma} = \frac{\gamma (n - 2)(n + 1)}{n^2}$$

The maximum gain is clearly increasing in $\gamma$. The gain is also increasing in $n$ because its first-order derivative with respect to $n$ is positive; that is, $\frac{\gamma}{4 - \gamma} \left(1 + \frac{4}{n^2}\right) > 0$. 