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How Does the Internet Affect the Financial Market? An Equilibrium Model of Internet-Facilitated Feedback Trading

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Appendix A

Proofs

Proof of Proposition 1: We first show that for all t, $\frac{1}{\lambda_t} \neq \beta_t$. Plugging Equations (3) and (6) into Equation (7) gives

$$dP_t = \lambda_t \beta_t dP_t + \lambda_t \sigma dW_t + \lambda_t \alpha_t (\tilde{v} - P_t) dt$$

If $\frac{1}{\lambda_t} = \beta_t$, then $\sigma dW_t = -\alpha_t (\tilde{v} - P_t) dt$ holds for all $\sigma > 0$ and $\alpha_t > 0$. Mathematically, it incorrectly implies that the Brownian motion is determined by a drift in time. From a practical point of view, it incorrectly implies that informed traders bring only noise into the market.

When $\frac{1}{\lambda_t} \neq \beta_t$, we have

$$dP_t = \tfrac{\lambda_t \alpha_t}{1-\lambda_t \beta_t} (\tilde{v} - P_t) dt + \tfrac{\lambda_t \sigma}{1-\lambda_t \beta_t} dW_t \tag{22}$$

Note that Equation (22) is under filtration of $F_t' = F_t \vee \sigma(\tilde{v})$. For a given F_t, taking the conditional expectation of Equation (22) yields

$$dP_t = \tfrac{\lambda_t \sigma}{1-\lambda_t \beta_t} dW_t \tag{23}$$

This is a stochastic differential equation of P_t under filtration F_t. To examine the properties of the price process, we need to apply the filtering lemma by Lipster and Shiryaev (1977), which helps answer the following question: Given the observations of the stochastic process P_t, what is the best estimate of the state $\tilde{v}$ based on these observations?

First, let $V_t = E(\tilde{v} | F_t)$ and consider the filtering of $\tilde{v}$ with respect to $\{F_t\}_{0 \le t \le 1}$. By the filtering lemma, we have

$$dV_t = \delta(t) \frac{\lambda_t \alpha_t}{1-\lambda_t \beta_t} \left(\frac{1-\lambda_t \beta_t}{\lambda_t \sigma} \right)^2 \frac{\lambda_t \sigma}{1-\lambda_t \beta_t} dW_t$$

thus

$$dV_t = \delta(t)\tfrac{\alpha_t}{\sigma}dW_t \tag{24}$$

where

$$\delta(t) \equiv E\left[(\tilde{v} - V_t)^2 \middle| F_t\right]$$

satisfies the following one-dimensional Riccati differential equation:

$$\frac{d\delta(t)}{dt} = -\delta(t)\left(\frac{\lambda_t\alpha_t}{1-\lambda_t\beta_t}\right)^2\left(\frac{1-\lambda_t\beta_t}{\lambda_t\sigma}\right)^2\delta(t)$$

That is,

$$\frac{d\delta(t)}{dt} = -\frac{\alpha_t^2}{\sigma^2}\delta^2(t)$$

with initial value

$$\delta(0) = \sigma_v^2$$

The solution to this equation is

$$\delta(t) = \left[\sigma_v^{-2} + \int_0^t \tfrac{\alpha_s^2}{\sigma^2}ds\right]^{-1} \tag{25}$$

Plugging Equation (25) into Equation (24) and using the semi-strong efficiency condition gives

$$dP_t = \tfrac{\alpha_t}{\sigma}\left[\sigma_v^{-2} + \int_0^t \tfrac{\alpha_s^2}{\sigma^2}ds\right]^{-1}dW_t \tag{26}$$

Comparing coefficients of Equation (23) and Equation (26) yields

$$\left[\sigma_v^{-2} + \int_0^t \tfrac{\alpha_s^2}{\sigma^2}ds\right]^{-1}\tfrac{\alpha_t}{\sigma} = \tfrac{\lambda_t\sigma}{1-\lambda_t\beta_t} \tag{27}$$

Thus,

$$\tfrac{\sigma^2}{\alpha_t}\left[\sigma_v^{-2} + \int_0^t \tfrac{\alpha_s^2}{\sigma^2}ds\right] = \tfrac{1}{\lambda_t} - \beta_t \tag{28}$$

Since α_t is strictly positive, we can see that when the market is semi-strong efficient the depth of the market $\frac{1}{\lambda_t}$ is always greater than β_t.

Equation (27) can be rewritten as

$$\frac{\left(\tfrac{\alpha_t}{\sigma}\right)^2}{\left[\sigma_v^{-2} + \int_0^t \tfrac{\alpha_s^2}{\sigma^2}ds\right]^2} = \left(\frac{\lambda_t\sigma}{1-\lambda_t\beta_t}\right)^2$$

Integrating the above equation with respect to *dt* yields

$$\alpha_t = \frac{\lambda_t\sigma^2}{\left(\sigma_v^2 - \int_0^t\left(\tfrac{\lambda_s\sigma}{1-\lambda_s\beta_s}\right)^2 ds\right)(1-\lambda_t\beta_t)} \tag{29}$$

Again, since α_t is strictly positive, it is easy to see that for all $t \in [0, 1]$

$$\sigma_v^2 - \int_0^t\left(\frac{\lambda_s\sigma}{1-\lambda_s\beta_s}\right)^2 ds > 0$$

Q.E.D.

Proof of Proposition 2: By Schwartz inequality and the constraint, we now that

$$\int_0^1 \frac{\lambda_s}{1-\lambda_s\beta_s} ds \le \left(\int_0^1 \left[\frac{\lambda_s}{1-\lambda_s\beta_s}\right]^2 ds\right)^{\frac{1}{2}} \le \frac{\sigma_v}{\sigma}$$

The equality holds if and only if for any $t \in [0, 1]$,

$$\frac{\lambda_t \sigma}{1-\lambda_t\beta_t} = \frac{\sigma_v}{\sigma} \tag{30}$$

Q.E.D.

Proof of Theorem 3: Equation (16) can be obtained directly from Equation (30). Equation (17) is obtained by plugging Equation (16) into Equation (29). Equations (27) and (25) combined yields Equation (18). Finally, Equation (19) is obtained by combining Equations (15) and (30).

Q.E.D.

Proof of Proposition 4: Under the assumption in our model, the profit earned by uninformed traders can be expressed by

$$E\left[(1-)\int_0^t (\tilde{v} - P_s) dX_U(s)\right]$$

and

$$\begin{aligned}
&E\left[(1-)\int_0^t (\tilde{v} - P_s) dX_U(s)\right] \\
&= E\left[(1-)\int_0^t (\tilde{v} - P_s)(\beta_s dP_s + \sigma dW_s)\right] \\
&= E\left[(1-)\int_0^t (\tilde{v} - P_s)\left(\frac{\beta_s\lambda_s\alpha_s}{1-\lambda_s\beta_s}(\tilde{v} - P_s)ds + \left(\sigma + \frac{\lambda_s\beta_s\sigma}{1-\lambda_s\beta_s}\right)dW_s\right)\right] \\
&= E\left[(1-)\int_0^t \frac{\beta_s\lambda_s\alpha_s}{1-\lambda_s\beta_s}(\tilde{v} - P_s)^2 ds\right] + E\left[(1-)\int_0^t (\tilde{v} - P_s)\frac{\sigma}{1-\lambda_s\beta_s} dW_s\right] \\
&= \int_0^t \frac{\beta_s\lambda_s\alpha_s}{1-\lambda_s\beta_s}\delta(s)ds + E\left[(1-)\int_0^t \left(\tilde{v} - \left(P_0 + \int_0^s \lambda_q dX_I(q) + \int_0^s \lambda_q dX_U(q)\right)\right)\frac{\sigma}{1-\lambda_s\beta_s} dW_s\right] \\
&= \sigma_v^2 \int_0^t \beta_s ds - E\left[(1-)\int_0^t \left(\int_0^s \lambda_q dX_U(q)\right)\frac{\sigma}{1-\lambda_s\beta_s} dW_s\right] \\
&= \sigma_v^2 \int_0^t \beta_s ds - E\left[(1-)\int_0^t \left(\int_0^s \frac{\beta_q\lambda_q^2\alpha_q}{1-\lambda_q\beta_q}(\tilde{v} - P_q)dq + \frac{\lambda_q\sigma}{1-\lambda_q\beta_q}dW_q\right)\frac{\sigma}{1-\lambda_s\beta_s} dW_s\right] \\
&= \sigma_v^2 \int_0^t \beta_s ds - E\left[(1-)\int_0^t \frac{\sigma\sigma_v}{1-\lambda_s\beta_s} W_s dW_s\right] \\
&= \sigma_v^2 \int_0^t \beta_s ds - \sigma_v^2 \int_0^t \left(\tfrac{\sigma}{\sigma_v} + \beta_s\right)ds \\
&= -\sigma_v \int_0^t \sigma ds \\
&= -\sigma_v \sigma t
\end{aligned}$$

The result is obtained from Equations (7), (22), (16), and (18), and the transformation relation between the Itô and the (1–) stochastic integration. The last equation assumes that σ is not a function of t. If σ is indeed a function of t, the result is not changed: $\int_0^t \sigma ds$ simply measures the average variance of noise up to time t. Whether σ is a function of time does not change any of our results.

Q.E.D.

Proof of Theorem 5: Without loss of generality, we suppose $P_0 = 0$. The second moment of the informed trader's profits is

$$E\left[(1-)\int_0^1(\tilde{v}-P_t)dX_I(t)\right]^2 = E\left[(1-)\int_0^1(\tilde{v}-P_t)^2\cdot\frac{\sigma}{\sigma_v}\cdot\frac{1}{1-t}dt\right]^2$$
$$= E\left[\int_0^1\left(\tilde{v}-\tilde{v}t-(1-t)\int_0^t\frac{\sigma_v}{1-s}dW_s\right)^2\cdot\frac{\sigma}{\sigma_v}\cdot\frac{1}{1-t}dt\right]^2$$
$$= E\left[\int_0^1(1-t)\left(\tilde{v}-\int_0^t\frac{\sigma_v}{1-s}dW_s\right)^2\cdot\frac{\sigma}{\sigma_v}dt\right]^2$$
$$= E\left[\int_0^1(1-t)\tilde{v}^2\frac{\sigma}{\sigma_v}dt-2\tilde{v}\int_0^1(1-t)\frac{\sigma}{\sigma_v}\left(\int_0^1\frac{\sigma_v}{1-s}dW_s\right)dt+\int_0^1(1-t)\left(\int_0^t\frac{\sigma_v}{1-s}dW_s\right)^2\frac{\sigma}{\sigma_v}dt\right]^2$$

Defining the first term by A_1,

$$A_1 \equiv \int_0^1(1-t)\tilde{v}^2\frac{\sigma}{\sigma_v}dt = \frac{\sigma}{2\sigma_v}\tilde{v}^2$$

Integrating by parts (stochastic integration, generalized Itô formula), we can have

$$\int_0^t\frac{1}{1-s}dW_s = \frac{W_s}{1-s}\Bigg|_0^t-\int_0^t W_s d\left(\frac{1}{1-s}\right) = \frac{W_t}{1-t}-\int_0^t\frac{W_s}{(1-s)^2}ds$$

By interchangeability of ordinary Riemann integration, we can calculate

$$A_2 \equiv \int_0^1(1-t)2\tilde{v}\left(\int_0^t\frac{\sigma_v}{1-s}dW_s\right)\frac{\sigma}{\sigma_v}dt$$
$$= 2\tilde{v}\sigma\int_0^1(1-t)\left(\int_0^t\frac{1}{1-s}dW_s\right)dt$$
$$= 2\tilde{v}\sigma\int_0^1(1-t)\left(\frac{W_t}{1-t}-\int_0^t\frac{W_s}{(1-s)^2}ds\right)dt$$
$$= 2\tilde{v}\sigma\left[\int_0^1 W_t dt-\int_0^1\int_0^t(1-t)\frac{W_s}{(1-s)^2}dsdt\right]$$
$$= 2\tilde{v}\sigma\left[\int_0^1 W_t\left(1-\frac{1}{1-t}+\frac{1}{2}\cdot\frac{1+t}{1-t}\right)dt\right]$$
$$= \tilde{v}\sigma\int_0^1 W_t dt$$

And the last term

$$A_3 \equiv \int_0^1(1-t)\cdot\frac{\sigma}{\sigma_v}\left(\int_0^t\frac{\sigma_v}{1-s}dW_s\right)^2dt$$
$$= \sigma\sigma_v\int_0^1(1-t)\left(\int_0^t\frac{1}{1-s}dW_s\right)^2dt$$

The informed trader's variance of the profit is

$$
\begin{aligned}
& E\left[A_1 + A_2 + A_3\right]^2 - \left(E\left[\tilde{\pi}(1)\right]\right)^2 \\
&= E\left[A_1^2 + A_2^2 + A_3^2 + 2A_1A_2 + A_1A_3 + 2A_2A_3\right] - \left(E\left[\tilde{\pi}(1)\right]\right)^2 \\
&= \frac{\sigma^2}{4\sigma_v^2} E\left[\tilde{v}^4\right] + \sigma^2 E\left[\tilde{v}^2\right] E\left[\int_0^1 W_t dt\right]^2 + \sigma_v^2 \sigma^2 E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s} dW_s\right)^2 dt\right]^2 + \frac{1}{2}\sigma^2 E\left[\tilde{v}^2\right] - \sigma_v^2 \sigma^2 \\
&= \sigma_v^2 \sigma^2 \left[\frac{1}{4} + E\left[\int_0^1 W_t dt\right]^2 + E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s} dW_s\right)^2 dt\right]^2 + \frac{1}{2} - 1\right] \\
&= \frac{1}{4}\sigma_v^2 \sigma^2
\end{aligned}
$$

We have used the assumption that $\tilde{v}$ is independent of the Brownian motion W_t, and the expectation of $\tilde{v}$ is zero (i.e., $\overline{P} \equiv E\left[\tilde{v}\right] = 0$), and the last equality is obtained from results in Appendix B.

We continue to calculate the variance of the uninformed traders' profits. For simplicity, we suppose that β_t is a constant over t, denoted by β, the second moment of the uninformed traders' profit is

$$
\begin{aligned}
& E\left[(1-)\int_0^1 (\tilde{v} - P_t) dX_U(t)\right]^2 \\
&= E\left[(1-)\int_0^1 (\tilde{v} - P_t) \cdot (\beta dP_t + \sigma dW_t)\right]^2 \\
&= E\left[(1-)\int_0^1 (\tilde{v} - P_t) \cdot \beta dP_t + (1-)\int_0^1 (\tilde{v} - P_t) \sigma dW_t\right]^2 \\
&= E\left[(1-)\int_0^1 (\tilde{v} - P_t)^2 \beta \cdot \frac{1}{1-t} dt + (1-)\int_0^1 (\tilde{v} - P_t) \cdot (\beta\sigma_v + \sigma) dW_t\right]^2 \\
&= E\left[\int_0^1 \frac{\beta}{1-t} (\tilde{v} - P_t)^2 dt\right]^2 + 2E\left[\left(\int_0^1 \frac{\beta}{1-t} (\tilde{v} - P_t)^2 dt\right)(1-)\int_0^1 (\tilde{v} - P_t) \cdot (\beta\sigma_v + \sigma) dW_t\right] \\
&\quad + E\left[(1-)\int_0^1 (\tilde{v} - P_t) \cdot (\beta\sigma_v + \sigma) dW_t\right]^2
\end{aligned}
$$

The first term

$$
\begin{aligned}
B_1 &\equiv E\left[\beta^2 \left(\int_0^1 \frac{1}{1-t} (\tilde{v} - P_t)^2 dt\right)^2\right] \\
&= \beta^2 \cdot \frac{\sigma_v^2}{\sigma^2} \sigma_v^2 \sigma^2 \left[\frac{5}{4} + E\left[\int_0^1 W_t dt\right]^2 + E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s} dW_s\right)^2 dt\right]^2\right] \\
&= \beta^2 \sigma_v^4 \left[\frac{5}{4} + E\left[\int_0^1 W_t dt\right]^2 + E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s} dW_s\right)^2 dt\right]^2\right] \\
&= \frac{7}{4}\beta^2 \sigma_v^4
\end{aligned}
$$

This last equality is obtained from Appendix B.

The second term

$$
\begin{aligned}
&B_2 \equiv 2\beta(\beta\sigma_v + \sigma)E\left[\left(\int_0^1 \frac{1}{1-t}(\tilde{v} - P_t)^2 dt\right)(1-)\int_0^1 (\tilde{v} - P_t)dW_t\right] \\
&= 2\beta(\beta\sigma_v + \sigma)\frac{\sigma_v}{\sigma}E\left[(A_1 + A_2 + A_3)\left(\int_0^1 (\tilde{v} - P_t)dW_t + \int_0^1 -\sigma_v dt\right)\right] \\
&= 2\beta(\beta\sigma_v + \sigma)\frac{\sigma_v}{\sigma}E\left[(A_1 + A_2 + A_3)\left(\int_0^1 (1-t)\left(\tilde{v} - \int_0^1 \frac{\sigma_v}{1-s}dW_s\right)dW_t - \sigma_v\right)\right] \\
&= 2\beta(\beta\sigma_v + \sigma)\frac{\sigma_v}{\sigma}\left[E\left[(A_1 + A_2 + A_3)\tilde{v}\int_0^1 (1-t)dW_t\right] - E\left((A_1 + A_2 + A_3)\int_0^1 (1-t)\int_0^t \frac{\sigma_v}{1-s}dW_s dW_t\right) - E(A_1 + A_2 + A_3)\sigma_v\right] \\
&= 2\beta(\beta\sigma_v + \sigma)\frac{\sigma_v}{\sigma}\left[\sigma_v^2\sigma E\left(\int_0^1 W_t dt\right)^2 - \sigma_v E\left((A_1 + A_2 + A_3)\int_0^1 (1-t)\int_0^t \frac{1}{1-s}dW_s dW_t - 1\right)\right] \\
&= 2\beta(\beta\sigma_v + \sigma)\frac{\sigma_v}{\sigma}\left[\sigma_v^2\sigma E\left(\int_0^1 W_t dt\right)^2 - \sigma_v^2\sigma E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s}dW_s\right)^2 dt \cdot \int_0^1 (1-t)\int_0^t \frac{1}{1-s}dW_s dW_t\right] - \frac{7}{4}\sigma_v^2\sigma\right] \\
&= 2\beta\sigma_v^3(\beta\sigma_v + \sigma)\left[E\left(\int_0^1 W_t dt\right)^2 - E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s}dW_s\right)^2 dt \cdot \int_0^1 (1-t)\int_0^t \frac{1}{1-s}dW_s dW_t\right] - \frac{7}{4}\right] \\
&= -3\beta\sigma_v^3(\beta\sigma_v + \sigma)
\end{aligned}
$$

The third term

$$
\begin{aligned}
B_3 &\equiv (\beta\sigma_v + \sigma)^2 E\left[\left((1-)\int_0^1 (\tilde{v} - P_t)dW_t\right)^2\right] \\
&= (\beta\sigma_v + \sigma)^2 E\left[\left(\int_0^1 (\tilde{v} - P_t)dW_t + \int_0^1 -\sigma_v dt\right)^2\right] \\
&= (\beta\sigma_v + \sigma)^2\left[\int_0^1 E(\tilde{v} - P_t)^2 dt + \sigma_v^2\right] \\
&= \frac{3}{2}\sigma_v^2(\beta\sigma_v + \sigma)^2
\end{aligned}
$$

Here we have used the isometric property of the stochastic integral. The uninformed traders' variance of profits is, therefore,

$$
\begin{aligned}
&B_1 + B_2 + B_3 - (\sigma^2\sigma_v^2) = \frac{7}{4}\beta^2\sigma_v^4 - 3\beta\sigma_v^3(\beta\sigma_v + \sigma) + \frac{3}{2}\sigma_v^2(\beta\sigma_v + \sigma)^2 - (\sigma^2\sigma_v^2) \\
&= \tfrac{1}{4}\beta^2\sigma_v^4 + \frac{1}{2}\sigma^2\sigma_v^2
\end{aligned}
$$

Q.E.D.

Reference

Lipster, R., and Shiryaev, A. 1977. *Statistics of Random Processes*, Berlin: Springer-Verlag.

Appendix B

Calculation of Expectations

Here we show how to calculate some expectations useful for Appendix A. First, we calculate $E\left[\int_0^1 W_t dt\right]^2$:

$$E\left[\int_0^1 W_t dt\right]^2 = E\left[\int_0^1 W_t dt \int_0^1 W_s ds\right]$$
$$= \int_0^1\int_0^1 E\left[W_t W_s\right] dtds$$
$$= \int_0^1\int_0^1 min(s,t) dtds$$
$$= \int_0^1\left(\int_0^s tdt + \int_s^1 sdt\right)ds$$
$$= \int_0^1\left(\frac{s^2}{2} + s(1-s)\right)ds = \frac{1}{3}$$

In the following, we calculate

$$\Gamma \equiv E\left[\int_0^1 (1-t)\left(\int_0^1 \frac{1}{1-s} dW_s\right)^2 dt \cdot \int_0^1 (1-t)\int_0^t \frac{1}{1-s} dW_s dW_t\right]$$

Let

$$I_t = \int_0^t \frac{1}{1-s} dW_s$$
$$X_\tau = \int_0^\tau (1-t) I_t^2 dt$$
$$Y_\tau = \int_0^\tau (1-t) I_t dW_t$$

where I_t and Y_τ are martingales. What we want is

$$X_1 Y_1$$

Integrating by parts,

$$X_1 Y_1 = \int_0^1 X_\tau dY_\tau + \int_0^1 Y_\tau dX_\tau$$

Since Y_τ is a martingale,

$$E\left[X_1 Y_1\right] = E\left[\int_0^1 Y_\tau dX_\tau\right]$$
$$= E\left[\int_0^1 Y_\tau (1-\tau) I_\tau^2 d\tau\right]$$
$$= \int_0^1 (1-\tau) E\left[Y_\tau I_\tau^2\right] d\tau$$

Integrating by parts, we have

$$Y_\tau I_\tau^2 = \int_0^\tau I_t^2 dY_t + \int_0^\tau Y_t dI_t^2 + \frac{1}{2}\int_0^\tau d < Y, I^2 >_t$$

where $<X, Y>_t$ denotes the quadratic variation process of X and Y.

$$dI_t^2 = 2I_t dI_t + d < I, I >_t = 2I_t dI_t + \frac{1}{(1-t)^2} dt$$

therefore

$$E\left[Y_\tau I_\tau^2\right] = E\left[\int_0^\tau Y_t \frac{1}{(1-t)^2}dt + \frac{1}{2}\int_0^\tau (1-t)I_t \cdot 2I_t \frac{1}{1-t}dt\right]$$
$$= \int_0^\tau E\left[I_t^2\right]dt = \int_0^\tau \left(\int_0^t \frac{1}{(1-s)^2}ds\right)dt$$
$$= \int_0^\tau \left(\frac{1}{1-t} - 1\right)dt = -\ln(1-\tau) - \tau$$

$$E\left[X_1 Y_1\right] = \int_0^1 (1-\tau)E\left[Y_\tau I_\tau^2\right]d\tau$$
$$= -\int_0^1 x\ln(x)dx - \int_0^1 (1-\tau)\tau d\tau = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Hence we obtain

$$\Gamma = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

Next, we calculate $E\left[\int_0^1 (1-t)\left(\int_0^t \frac{1}{1-s}dW_s\right)^2 dt\right]^2$. Using the same notation as above, what we want is $E\left[X_1^2\right]$. Integrating by parts,

$$X_1 X_1 = 2\int_0^1 X_\tau dX_\tau$$

Hence,

$$E\left[X_1^2\right] = 2E\left[\int_0^1 X_\tau dX_\tau\right]$$
$$= 2E\left[\int_0^1 X_\tau (1-\tau)I_\tau^2 d\tau\right]$$
$$= 2\int_0^1 (1-\tau)E\left[X_\tau I_\tau^2\right]d\tau$$

Integrating by parts, we have

$$X_\tau I_\tau^2 = \int_0^\tau I_t^2 dX_t + \int_0^\tau X_t dI_t^2 + \frac{1}{2}\int_0^\tau d < X, I^2 >_t$$
$$= \int_0^\tau (1-t)I_t^4 dt + \int_0^\tau X_t dI_t^2$$

where $<X, Y>_t$ denotes the quadratic variation process of X and Y.

$$dI_t^2 = 2I_t dI_t + d < I, I >_t = 2I_t dI_t + \frac{1}{(1-t)^2}dt$$

therefore

$$E\left[X_\tau I_\tau^2\right] = E\left[\int_0^\tau (1-t)I_t^4 dt + \int_0^\tau X_t \frac{1}{(1-t)^2}dt\right]$$
$$= \int_0^\tau (1-t)E\left[I_t^4\right]dt + \int_0^\tau E\left[X_t\right]\frac{1}{(1-t)^2}dt$$
$$= \int_0^\tau (1-t)\frac{3t^2}{(1-t)^2}dt + \frac{1}{2}\int_0^\tau \frac{t^2}{(1-t)^2}dt$$
$$= -3\left(\tau + \tfrac{\tau^2}{2}\right) - 3\ln(1-\tau) + \frac{1}{2}\left[\tau + \ln(1-\tau) + \frac{1}{1-\tau} - 1\right]$$
$$= -\frac{5}{2}\left(\tau + \ln(1-\tau)\right) + \frac{1}{2}\left[\frac{1}{1-\tau} - 1 - 3\tau^2\right]$$

Here we have used the result that $I_t \sim N\left(0, \frac{t}{1-t}\right)$, then $E\left[I_t^4\right] = \frac{3t^2}{[1-t]^2}$ and $E\left[I_t\right] = \frac{t^2}{2}$.

$$\begin{aligned} E\left[X_1^2\right] &= 2\int_0^1 (1-\tau) E\left[X_\tau I_\tau^2\right] d\tau \\ &= -\int_0^1 5x\ln(x)dx - 5\int_0^1 (1-\tau)\tau d\tau + \int_0^1 \tau d\tau - \int_0^1 (1-t)dt - \int_0^1 3\tau^2(1-\tau)d\tau \\ &= \frac{5}{4} - \frac{5}{6} - 1 + \frac{3}{4} = \frac{1}{6} \end{aligned}$$

Appendix C

Proof of Theorem 7

Proof of Theorem 7. Inserting the uninformed traders' new demand Equation (21) to the pricing rule Equation (7), we can obtain

$$dP_t = \left[\frac{\lambda_t \alpha_t}{1-\lambda_t \beta_t}\tilde{v} + \frac{\lambda_t \gamma_t}{1-\lambda_t \beta_t}(\tilde{v}+\varepsilon) - \frac{\lambda_t(\alpha_t + \gamma_t)}{1-\lambda_t \beta_t}P_t\right]dt + \frac{\lambda_t \sigma}{1-\lambda_t \beta_t}dW_t$$

The logic of deriving the result is the same as before. However, since we have one additional dimension of uncertainty coming from ε, the filtering process needs to deal with the vector $\begin{pmatrix}\tilde{v} \\ \tilde{v}+\varepsilon\end{pmatrix}$.

Consequently, the deviation of the price from the liquidation value at time t, $\delta(t)$, is a matrix

$$\delta(t) \equiv \begin{pmatrix}\delta_{11}(t) & \delta_{12}(t) \\ \delta_{21}(t) & \delta_{22}(t)\end{pmatrix}$$

with $\delta_{11}(t) = E\left[\tilde{v} - E\left(\tilde{v}|F_t\right)\right]^2$, $\delta_{12}(t) = \delta_{21}(t) = E\left[\left(\tilde{v} - E\left(\tilde{v}|F_t\right)\right)\left((\tilde{v}+\varepsilon) - E\left((\tilde{v}+\varepsilon)|F_t\right)\right)\right]$, and $\delta_{22}(t) = E\left[(\tilde{v}+\varepsilon) - E\left((\tilde{v}+\varepsilon)|F_t\right)\right]^2$.

The variance-covariance matrix can be derived as

$$d\begin{pmatrix}\delta_{11}(t) & \delta_{12}(t) \\ \delta_{21}(t) & \delta_{22}(t)\end{pmatrix} = -\begin{pmatrix}\delta_{11}(t) & \delta_{12}(t) \\ \delta_{21}(t) & \delta_{22}(t)\end{pmatrix}\begin{pmatrix}\frac{\alpha_t^2}{\sigma^2} & \frac{\alpha_t\gamma_t}{\sigma^2} \\ \frac{\alpha_t\gamma_t}{\sigma^2} & \frac{\gamma_t^2}{\sigma^2}\end{pmatrix}\begin{pmatrix}\delta_{11}(t) & \delta_{12}(t) \\ \delta_{21}(t) & \delta_{22}(t)\end{pmatrix}' dt \tag{31}$$

where the symbol $'$ denotes the transpose of the matrix.

Equation (31) is a matrix Riccati differential equation with initial value

$$\delta(0) = \begin{pmatrix}\delta_{11}(0) & \delta_{12}(0) \\ \delta_{21}(0) & \delta_{22}(0)\end{pmatrix} = \begin{pmatrix}\sigma_v^2 & \sigma_v^2 \\ \sigma_v^2 & \sigma_v^2 + \sigma_\varepsilon^2\end{pmatrix}$$

The solution to the equation is

$$\delta(t) = \left[\left(\delta(0)\right)^{-1} + \begin{pmatrix}\int_0^t \frac{\alpha_s^2}{\sigma^2}ds & \int_0^t \frac{\alpha_s\gamma_s}{\sigma^2}ds \\ \int_0^t \frac{\alpha_s\gamma_s}{\sigma^2}ds & \int_0^t \frac{\gamma_s^2}{\sigma^2}ds\end{pmatrix}\right]^{-1} \tag{32}$$

After calculation, we obtain that

$$\left(\delta(0)\right)^{-1} = \begin{pmatrix}\frac{1}{\sigma_v^2} + \frac{1}{\sigma_\varepsilon^2} & -\frac{1}{\sigma_\varepsilon^2} \\ -\frac{1}{\sigma_\varepsilon^2} & \frac{1}{\sigma_\varepsilon^2}\end{pmatrix}$$

and

$$\delta_{11}(t) = \frac{\left(\frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\gamma_s^2}{\sigma^2} ds\right)}{\left(\frac{1}{\sigma_v^2} + \frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\alpha_s^2}{\sigma^2} ds\right)\left(\frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\gamma_s^2}{\sigma^2} ds\right) - \left[\int_0^t \frac{\alpha_s\gamma_s}{\sigma^2} ds - \frac{1}{\sigma_\varepsilon^2}\right]^2}$$

$$\delta_{12}(t) = \delta_{21}(t) = \frac{-\left(\int_0^t \frac{\alpha_s\gamma_s}{\sigma^2} ds - \frac{1}{\sigma_\varepsilon^2}\right)}{\left(\frac{1}{\sigma_v^2} + \frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\alpha_s^2}{\sigma^2} ds\right)\left(\frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\gamma_s^2}{\sigma^2} ds\right) - \left[\int_0^t \frac{\alpha_s\gamma_s}{\sigma^2} ds - \frac{1}{\sigma_\varepsilon^2}\right]^2}$$

$$\delta_{22}(t) = \frac{\left(\frac{1}{\sigma_v^2} + \frac{1}{\sigma_\varepsilon^2} + \int_0^2 \frac{\alpha_s^2}{\sigma^2} ds\right)}{\left(\frac{1}{\sigma_v^2} + \frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\alpha_s^2}{\sigma^2} ds\right)\left(\frac{1}{\sigma_\varepsilon^2} + \int_0^t \frac{\gamma_s^2}{\sigma^2} ds\right) - \left[\int_0^t \frac{\alpha_s\gamma_s}{\sigma^2} ds - \frac{1}{\sigma_\varepsilon^2}\right]^2}$$

The informed trader's expected profit at time 1 is

$$E\left[\int_0^1 (\tilde{v} - P_t) dX_I(t)\right] = E\left[\int_0^1 \alpha_t (\tilde{v} - P_t)^2 dt\right] = \int_0^1 \alpha_t E\left[(\tilde{v} - P_t)^2\right] dt$$
$$= \int_0^1 \alpha_t \delta_{11}(t) dt$$

Same as before, the informed trader chooses α_t to maximize her expected profit. That is,

$$\alpha_t^* = arg\ max \int_0^1 \alpha_t \delta_{11}(t) dt$$

Same as $\delta(t)$ in the baseline model, $\delta_{11}(t)$ does not involve the feedback parameter β_t. So the informed trader's maximization problem is independent of the feedback intensity.

Similarly, the uninformed trader with imprecise information tries to maximize her profit at time 1. The maximization problem is

$$\gamma_t^* = arg\ max \int_0^1 \gamma_t \delta_{12}(t) dt$$

Since $\delta_{21}(t)$ does not involve the feedback parameter β_t, the optimal γ_t is also independent from the feedback intensity.

For the conditional expectation, we have

$$d\begin{pmatrix} E[\tilde{v}|F_t] \\ E[\tilde{v} + \varepsilon|F_t] \end{pmatrix} = \delta(t)\begin{pmatrix} \frac{\alpha_t}{\sigma^2} \\ \frac{\gamma_t}{\sigma} \end{pmatrix} dW_t + \delta(t)\begin{pmatrix} \frac{\alpha_t^2}{\sigma^2} & \frac{\alpha_t\gamma_t}{\sigma^2} \\ \frac{\alpha_t\gamma_t}{\sigma^2} & \frac{\gamma_t^2}{\sigma^2} \end{pmatrix}\begin{pmatrix} \tilde{v} - E[\tilde{v}|F_t] \\ (\tilde{v} + \varepsilon) - E[\tilde{v} + \varepsilon|F_t] \end{pmatrix} dt \tag{33}$$

Applying the semi-strong market-efficiency conditions $E[\tilde{v}|F_t] = P_t$, we have for all α_t and γ_t

$$\frac{\alpha_t}{\sigma}\delta_{11}(t) + \frac{\gamma_t}{\sigma}\delta_{12}(t) = \frac{\lambda_t \sigma}{1 - \lambda_t \beta_t} \tag{34}$$

Inserting the optimal values α_t^* and γ_t^*, we get the result about λ_t.

$$\lambda_t = \frac{\alpha_t^*\delta_{11}(t) + \gamma_t^*\delta_{12}(t)}{\sigma + \beta_t\left(\alpha_t^*\delta_{11}(t) + \gamma_t^*\delta_{12}(t)\right)} \tag{35}$$

This result is highly consistent with what we have obtained in **Theorem 3**. The expression of λ_t is very similar to that in the baseline model. The only difference is that the variance of the liquidation value in the baseline model is replaced by the variance and covariance of the liquidation value together with the error.

Overall, this completes the proof that, even if uninformed traders can obtain imprecise signals of information, feedback trading does not affect informed trader's strategy nor the market price process.

Q.E.D.